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The existence of solution for $p(x)$-Laplacian equation with no flux boundary [1]

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Abstract Using the principle least action, we investigate the existence solution for $p(x)$-Lapla -cian equation with no flux boundary condition in a bounded domain $\Omega \subset R^N$, where no flux boundary condition is given in the following:

$$\begin{cases} u = constant, & x \in \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases}$$

Keywords Variable exponent Sobolev spaces, $p(x)$-Laplacian, no flux boundary, the princi -ple least action.

§1.Introduction

In recent years there has been an increasing interest in the study of various mathematics problem with variable exponent, see the papers [1,2,3,5]. The existence of solutions of $p(x)$-Laplacian Dirichlet problems has been studied in many papers (see e.g. [7, 8, 9, 12, 14, 15]). The aim of the present paper is to study the existence of solutions of $p(x)-$Laplacian equation with no flux boundary. Where no flux boundary condition is given in the following:

$$\begin{cases} u = constant, & x \in \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases}$$

Throughout the paper, Ω will be bounded domain in R^N, $p \in C(\bar{\Omega})$ and

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p^+ := \sup_{x\in\Omega} p(x) < \infty.$$

Consider the existence of solution of the following problem

$$\begin{cases} -\triangle_{p(x)} u + f(x,u) = 0, & x \in \Omega, \\ u(x) = constant, & x \in \Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} ds = 0. \end{cases} \tag{1}$$

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When a term of $|u|^{p(x)-2}u$ is involved in $p(x)-$Laplacian equation, we can apply variation method to obtain the existence of solution and multiplicity easily. In this paper, we may use the principle least action to get the existence of solution when f satisfies some appropriate conditions.

§2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$. On the basic properties of the space $W^{1,p(x)}(\Omega)$ we refer to [4, 13]. In the following we display some facts which we will use later.

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω, and elements in $S(\Omega)$ that equal to each other almost everywhere are considered as one element. Denote $L^\infty_+(\Omega) = \{p \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega \mathrm{p(x)} := \mathrm{p}_- \geq 1\}$.

For $p \in L^\infty_+(\Omega)$, define

$$L^{p(x)}(\Omega) = \{u \in S(\Omega) : \int_\Omega |u|^{p(x)} \mathrm{d}x < \infty\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u/\lambda|^{p(x)} \mathrm{d}x \leq 1\};$$

and define

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Define

$$p^* = \begin{cases} \frac{Np(x)}{N-p(x)}, & if\ p(x) < N, \\ \infty, & if\ p(x) \geq N. \end{cases}$$

Hereafter, we always assume that $p(x)$ is continuous and $p_- > 1$.

Proposition 2.1.[4,13] The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, all are separable and reflexive Banach spaces.

Proposition 2.2.[4,10,13] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^o(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^o(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^o(x)}(\Omega)$, the Hölder inequality holds:

$$\int_\Omega |uv| \mathrm{d}x \leq 2|u|_{p(x)}|v|_{p^o(x)}. \tag{2}$$

Remark 2.3. In the right of (2), the constants 2 is suitable, but not the best. The best constant is given in [10] denoted by $d_{(p_-,p_+)}$ which only depends on p_- and p_+ when $p(x)$ is given and $d_{(p_-,p_+)}$ is smaller than $\frac{1}{p_-} + \frac{1}{p_+}$.

Proposition 2.4.[13] (Theorem 1.3.) Set $\rho(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x$. For $u, u_k \in L^{p(x)}(\Omega)$, we have:

(i) $|u|_{p(x)} < 1\ (= 1; > 1) \iff \rho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;

(iii) $\lim_{k\to\infty} |u_k|_{p(x)} = 0\ (= \infty) \iff \lim_{k\to\infty} \rho(u_k) = 0\ (= \infty)$.

Proposition 2.5.[11] (Theorem 1.1.) If p: $\Omega \to \mathbb{R}$ is Lipschitz continuous and $p_+ < N$, then for $q \in L_+^\infty(\Omega)$ with $p(x) \le q(x) \le p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.6.[6] (Proposition 2.4.) Ω is bounded. Assume that the boundary of Ω possesses the cone property and $p \in C(\bar{\Omega})$. If $q \in C(\bar{\Omega})$ and $1 \le q(x) < p^*(x)$ for $x \in \bar{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In this paper we use space $X := \{u \in W^{1,p(x)}(\Omega) : u|_{\partial\Omega} = \text{constant}\} = \mathrm{W}_0^{1,\mathrm{p(x)}}(\Omega) \oplus \mathrm{R}$. X is a subspace of $W^{1,p(x)}(\Omega)$. The space X is also separable and reflexive Banach space and with the equivalent norm

$$\|u\| = \|u\|_X = |\bar{u}| + |\nabla \tilde{u}|_{p(x)}, u \in X.$$

where $u = \bar{u} + \tilde{u}, \bar{u} \in R, \tilde{u} \in W_0^{1,p(x)}(\Omega)$.

Definition 2.7. We call $u \in X$ is a weak solution of (1), if u satisfies

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx = \int_\Omega f(x,u) v dx,\ \forall v \in X.$$

In this paper, we always suppose f satisfies the following basic assumption:

Basic assumption. Suppose f satisfies *Carathéodory*, and

$$(F_0)\quad |f(x,t)| \le \alpha + \beta |t|^{q(x)-1}, 1 < q(x) < P^*(x).$$

Consider the following function:

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega F(x,u) dx, u \in X,$$

where $F(x,t) = \int_0^t f(x,s) ds$. We obtain $I \in C^1(X, R)$.

The main result of this paper is given by the following theorem.

Theorem 2.8. Suppose (F_0) holds, I has a bounded minimizing sequence. Then I has a minimizer.

Proof. Set $\{u_n\}$ is a minimizing sequence of I, namely, $I(u_n) \to \inf_{n\to\infty} I(u)$, $\|u_n\|$ is bounded. Suppose $u_n \rightharpoonup u_0$ in X, obviously, I is weakly lower semicontinuous, thus we have $\inf_{u\in X} = \lim_{n\to\infty} I(u_n) \ge I(u_0)$, it follows that $I(u_0) = \inf_{u\in X} I(u)$.

§3. Main result

Theorem 3.1. Suppose $\alpha \in R$, and $0 \le \alpha < p^- - 1$, for any $x \in \Omega$, $t \in R$. And the following assumptions hold:

$$|f(x,t)| \le C_1 + C_2 |t|^\alpha. \tag{3}$$

$$|t|^{-\frac{\alpha p^-}{p^- - 1}} \int_\Omega F(x,t) dx \to +\infty \quad if\ |t| \to \infty. \tag{4}$$

Then I has a minimizer.

Proof. Clearly, I is weakly lower semicontinuous. We shall prove I is coercive.

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega (F(x,u(x)) - F(x,\bar{u}))dx + \int_\Omega F(x,\bar{u})dx.$$

where

$$\begin{aligned}\int_\Omega (F(x,u(x)) - F(x,\bar{u}))dx &= |\int_\Omega (\int_0^1 f(x,\bar{u}+s\tilde{u}(x))\tilde{u}(x)ds)dx| \\ &\leq \int_\Omega (C_1 + C_3|\bar{u}|^\alpha + C_3|\tilde{u}(x)|^\alpha)|\tilde{u}(x)|dx \\ &\leq \int_\Omega C_1|\tilde{u}(x)|dx + \int_\Omega C_3|\bar{u}|^\alpha|\tilde{u}(x)|dx + \int_\Omega C_3|\tilde{u}(x)|^{\alpha+1}dx \\ &\triangleq \mathbf{I} + \mathbf{II} + \mathbf{III}.\end{aligned}$$

We next estimate forms $\mathbf{I}, \mathbf{II}, \mathbf{III}$. Where

$$\mathbf{I} = C_4|\tilde{u}|_{L^1(\Omega)} \leq C_5|\tilde{u}|_{L^{p(x)}(\Omega)} \leq C_6|\nabla\tilde{u}|_{L^{p(x)}(\Omega)}. \tag{5}$$

$$\mathbf{III} = C_3|\tilde{u}|^{\alpha+1}_{L^{\alpha+1}} \leq C_7|\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)} \leq C_8|\nabla\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)}. \tag{6}$$

According to Young inequality, we deduce

$$\begin{aligned}\mathbf{II} &= C_3\int_\Omega |\bar{u}|^\alpha|\tilde{u}(x)|dx \\ &\leq C_3\int_\Omega (\varepsilon|\tilde{u}(x)|^{p^-} + C_9(\varepsilon)|\bar{u}|^{\frac{\alpha p^-}{p^- -1}})dx \\ &\leq \varepsilon C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} + C_3C_9(\varepsilon)|\Omega||\bar{u}|^{\frac{\alpha p^-}{p^- -1}}.\end{aligned}$$

Fix $\varepsilon > 0$ so small that $\varepsilon C_{10} \leq \frac{1}{2p^+}$. we conclude that

$$\mathbf{II} \leq \frac{1}{2p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} + C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- -1}}. \tag{7}$$

By (3), (4) and (5),we have

$$\begin{aligned}I(u) &\geq \frac{1}{p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - \frac{1}{p^+} - C_6|\nabla\tilde{u}|_{L^{p(x)}(\Omega)} - \frac{1}{2p^+}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- -1}} \\ &\quad - C_8|\nabla\tilde{u}|^{\alpha+1}_{L^{p(x)}(\Omega)} + \int_\Omega F(x,\bar{u})dx \\ &\geq C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{12} + \int_\Omega F(x,\bar{u})dx - C_{11}|\bar{u}|^{\frac{\alpha p^-}{p^- -1}} \\ &\geq C_{10}|\nabla\tilde{u}|^{p^-}_{L^{p(x)}(\Omega)} - C_{12} + |\bar{u}|^{\frac{\alpha p^-}{p^- -1}}(|\bar{u}|^{\frac{\alpha p^-}{p^- -1}}(\int_\Omega F(x,\bar{u})dx - C_{11})).\end{aligned} \tag{8}$$

Using (1) and (6) we get I is coercive, thus I has a minimizer.

Given $\alpha = 0$ in theorem 3.1, we have

Corollary 3.2. Suppose f satisfies

$$|f(x,t)| \leq C_1. \tag{9}$$

$$\int_\Omega F(x,t)dx \to \infty,\ if\ |t| \to \infty. \tag{10}$$

Then I has a minimizer.

Theorem 3.3. Suppose $F(x,t)$ is convex and $\int_\Omega F(x,t)dx \to \infty$ when $|t| \to \infty$. Then I has a minimizer.

Proof. Set $g(t) = \int_\Omega F(x,t)dx$, for any $t \in R$. Obviously $g : R \to R$ is a differentiable and convex function, and when $|t| \to \infty$, $g(t) \to +\infty$. We deduce g has a minimizer t_*, thus $g'(t_*) = 0$. Namely $\int_\Omega f(x,t_*)dx = 0$. Suppose $\{u_n\}$ is a minimizing sequence of I, according to the proposition of $F(x,t)$, we have

$$\begin{aligned} I(u_n) &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx + \int_\Omega (F(x,t_*) + f(x,t_*))(u_n(x) - t_*)dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx - C_1 + \int_\Omega f(x,t_*)\tilde{u}_n(x)dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla \tilde{u}_n|^{p(x)}dx - C_1 - C_2|\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)}. \end{aligned} \tag{11}$$

We derive that $|\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)}$ is bounded from (9). Next we proof $|\bar{u}_n|_{L^{p(x)}(\Omega)}$ is also bounded. Owing now to the convexity of F, we deduce that

$$F(x, \frac{\bar{u}_n}{2}) = F(x, \frac{u_n(x) - \tilde{u}_n(x)}{2}) \leq \frac{1}{2}(F(x,u_n(x)) + F(x,-\tilde{u}_n(x))). \tag{12}$$

From (10), we have

$$\begin{aligned} I(u_n) &\geq \int_\Omega F(x,u_n)dx \geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - \int_\Omega F(X,-\tilde{u}_n(x))dx \\ &\geq 2\int_\Omega f(x,\frac{\bar{u}_n}{2})dx - \int_\Omega (C_{12} + C_{13}|\bar{u}_n(x)|^{q(x)})dx \\ &\geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - C_{12}|\Omega| - C_{14}|\nabla \tilde{u}_n|^{p^+}_{L^{p(x)}(\Omega)} - C_{15} \\ &\geq 2\int_\Omega F(x,\frac{\bar{u}_n}{2})dx - C_{16}. \end{aligned} \tag{13}$$

We get that $|\bar{u}_n|_{L^{p(x)}(\Omega)}$ is bounded, thus $\|u_n\|$ bounded. Consequently, I has a minimizer.

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